# B. Math. I Year
# First Semester 2000 - 2001
# Final Exam / Analysis
# Time: 2 Hours

1. a) Find $\lim\limits_{n\to\infty} \frac{e^n+5}{e^n+n^2}$.

   b) Find $\lim\limits_{n\to\infty} n \log\left(1+\frac{1}{n}\right)$. [5+5]

2. Decide if the following statements are true or false. If true, give a proof, if false, give a counter example:

   a) $a_n'$s are real numbers, such that $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Then $\sum\limits_{n=1}^{\infty} a_n^2$ is convergent.

   b) $a_n \geq 0$ and $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Then $\sum\limits_{n=1}^{\infty} a_n^2$ is convergent. [5+5]

3. If $|x| < 1$, prove that $\sum\limits_{n=1}^{\infty} n^3 x^{n^2}$ is convergent. [5]

4. (a) Complete the following sentence:
   $f : I \to \boldsymbol{R}, f$ is uniformly continuous on $I$ iff ........

   b) Let $I = [1, \infty)$ and $f(x) = \frac{1}{x}$. Prove that $f$ is uniformly continuous on $I$. [5+10]

5. $f$ is a $C^1$-function on $[1, \infty)$. $f(1) = 1$ and $f$ is increasing. If $f'(x) \leq \frac{1}{x^2}$, show that $\lim\limits_{x\to\infty} f(x)$ exists. [10]

6. Let $q_1, q_2, \ldots$ be an enumeration of the rationals on $[0, 1]$. Define $f$ on $[0, 1]$ by $f(q_n) = \frac{1}{n}$ and $f(x) = 1$ if $x$ is irrational.
   Is $f$ Riemann integrable on $[0, 1]$ ? Justify your answer. [10]

7. Let $f$ be a $C^1$-function on $\boldsymbol{R}$. $f(0) = 0$ and $f'(q) = 1$ if $q$ is rational. Find $f$. [10]

8. Let $f(x) = xe^{x^2}$. Find the second Taylor polynomial around $a = 0$. Using this find an approximate value of $f(0.01)$. Estimate the maximum error involved in this method. [20]

9. Let $f(x) = \int\limits_0^{e^x} e^{t^2} dt$. Find $f'(x)$. [10]